Heat transfer from surfaces of non-uniform temperature

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SUMMARY

The paper deals with the calculation of steady heat transfer from a surface of arbitrary temperature distribution to a laminar semi-infinite stream of arbitrary velocity distribution. Lighthill's method is improved by a correction which accounts for the departures from linearity of the velocity profile within the thermal boundary layer, and which comprehends the influences of Prandtl number, pressure gradient, body forces, and non-coincident start of velocity and thermal layers. Methods are given for evaluating the total heat flux directly, and for integrating the differential equations for the growth of the boundary layer thickness by means of quadratures.

1. INTRODUCTION

Approximate procedures for calculating the heat transfer from a body to a laminar stream flowing steadily around it fall into two classes. The first class contains methods which implicitly assume a fixed relation between the thermal and velocity boundary thicknesses. Such is the method of Eckert (1942), which has recently been simplified and extended (Smith & Spalding 1958; Spalding & Smith 1958). These class I procedures are valid only for uniform wall temperature.

Procedures of the second class permit the boundary layer thickness ratio to vary; an ordinary differential equation is set up for each thickness. Such are the methods of Squire (1942), Lighthill (1950), and Schuh (1953). Class II procedures may be used for arbitrary wall-temperature variation.

The method of Lighthill is asymptotically exact when the thermal boundary layer is much thinner than that of the velocity, as occurs, for example, when the front part of the body is at the same temperature as the stream. Even when the thicknesses are of the same order, the method is approximately correct provided that the velocity profile along a normal to the surface is nearly linear; this occurs when there is no longitudinal pressure gradient.

Tribus & Klein (1955) have attempted to improve the Lighthill method by introducing a correction for pressure gradient due to Tifford (1951). It will be shown that this procedure may actually impair the correctness of the calculation. The present paper describes an alternative correction which reduces the error of the Lighthill method to 2·5%, regardless of pressure gradient.

In addition to this improvement the present paper describes a rapid method of integrating the differential equations representing the growth of the thermal and velocity boundary-layer thicknesses, and also describes a procedure giving the total heat transfer rate.

It is convenient to consider the case of uniform wall temperature first. Cases of variable wall temperature are then dealt with by superposition. Finally the calculation of the shear force distribution is discussed.

2. Heat transfer from a surface of uniform temperature

2.1. *The problem*

Figure 1 illustrates an aerofoil. Measured above the free stream temperature, the wall temperature is zero up to a distance ξ from the leading edge, but at larger values of x it has the value T_0. The fluid velocity normal to the wall is supposed zero, and the fluid properties are taken as uniform. Figure 1 also illustrates the growth of the velocity and thermal boundary layers. It is seen that the latter is typically thinner than the former, at least where x is not much greater than ξ.

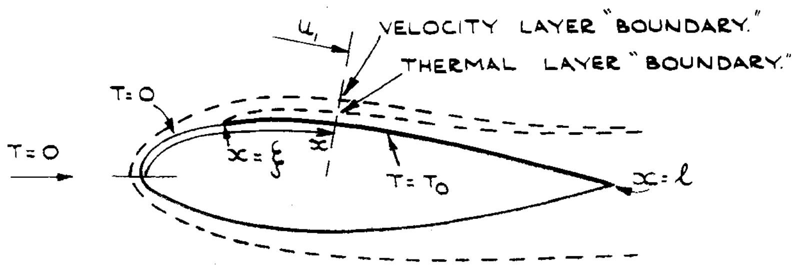

Figure 1. Velocity and thermal boundary layers on an aerofoil.

It is convenient to discuss the properties of the velocity and temperature distributions in terms of boundary-layer thicknesses. The nomenclature used conforms with that of Smith & Spalding (1958) and is summarized in table 1, in which u is the x-component of velocity, u_1 is the velocity outside the boundary layer, T is the fluid temperature, and y is the normal distance from the wall. Other notation will be introduced as required.

Momentum thickness	$\delta_2 \equiv \int_0^\infty \frac{u}{u_1}\left(1-\frac{u}{u_1}\right) dy$
Shear thickness	$\delta_4 \equiv u_1 \Big/ \left(\frac{\partial u}{\partial y}\right)_{y=0}$
Enthalpy flux thickness	$\Delta_2 \equiv \int_0^\infty \frac{uT}{u_1 T_0}\, dy$
Conduction thickness	$\Delta_4 \equiv -T_0 \Big/ \left(\frac{\partial T}{\partial y}\right)_{y=0}$

Table 1.

The problem is to determine the local and total heat fluxes from the wall. It may be considered solved when Δ_4 has been determined as a function of x; for the local heat flux $\dot{q}''$ can then be evaluated from the equation

$$\dot{q}'' = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = k\frac{T_0}{\Delta_4}, \tag{1}$$

where k is the thermal conductivity of the fluid, assumed uniform. If the total heat transfer per unit width of aerofoil, $\dot{q}'$, is required, $\dot{q}''$ can be integrated over the surface. Alternatively, it may be more convenient to calculate Δ_2 at the trailing edge ($x = l$) and then apply the steady-flow energy equation in the form

$$\dot{q}' = \int_0^l \dot{q}''\, dx = (c\rho u_1 T_0 \Delta_2)_{x=l}, \tag{2}$$

where c is the fluid specific heat, and ρ is the fluid density.

2.2. *Dimensional analysis*

As is common in the approximate solution of parabolic partial differential equations, a 'profile' method is used; that is, the velocity and temperature profiles along normals to the wall are assumed to belong to a restricted family, in this case those that are found in boundary layers adjacent to isothermal wedges in laminar steady flow. For the moment, the shear thickness δ_4 is supposed known as a function of x, and attention is concentrated on the conduction thickness Δ_4.

We assume that the rate of growth of Δ_4 in the x-direction depends only on the local values of stream velocity, velocity gradient, kinematic viscosity ν, thermal diffusivity α, and conduction thickness. Standard methods of dimensional analysis can be used to restrict the form of the relation between these quantities. In performing the analysis it is helpful to ascribe different dimensions to lengths in the x- and y-directions; for then, by using the boundary-layer assumption that viscosity and conductivity are responsive only to gradients in the y-direction, the following continuation equation can be directly derived:

$$\frac{u_1}{\nu}\frac{d\Delta_4^2}{dx} = f\left(\frac{\delta_4^2}{\nu}\frac{du_1}{dx}, \frac{\Delta_4}{\delta_4}, \sigma\right) \tag{3}$$

where σ is the Prandtl number ν/α.

This is a first-order differential equation, generally non-linear. The function on the right-hand side has yet to be determined. When this has been done, standard methods of numerical analysis will yield the dependent variable Δ_4 as a function of the independent variable x; for u_1 and δ_4 are already known as functions of x, while ν and σ are known constants.

2.3. *The Lighthill method*

Lighthill (1950) has considered the case in which the thermal boundary layer is so much thinner than that of velocity that it may be regarded as lying wholly within a region of linear velocity profile. To proceed as far as possible in the direction of Lighthill's result by means of dimensional analysis, it should be noted that in this case u_1 and δ_4 can enter equation (3) only in the form of the gradient u_1/δ_4, and that, further, ν must have no influence. If it is also noted that the application of dimensional analysis to the velocity boundary layer gives the continuation equation

$$u_1 \frac{d\delta_4^2}{dx} = f\left(\delta_4^2 \frac{du_1}{dx}\right), \tag{4}$$

where f is an arbitrary function, then the form of the function in (3) can be deduced. This deduction leads to the following compact continuation equation:

$$\frac{1}{\alpha}\left(\frac{\delta_4}{u_1}\right)^{1/2} \frac{d}{dx}\left[\Delta_4^3 \left(\frac{u_1}{\delta_4}\right)^{3/2}\right] = \text{const.} \tag{5}$$

That (5) is a special form of (3), combined with (4), can be verified by differentiation of the term within the square brackets.

By exact solution of the differential energy equation, Lighthill showed that the constant in (5) is 6·41. His result was expressed in the equivalent integral form of (5), which, in the present notation, becomes

$$\frac{1}{\Delta_4} = \left(\frac{1}{6{\cdot}41}\right)^{1/3} \frac{1}{\alpha^{1/3}} \left(\frac{u_1}{\delta_4}\right)^{1/2} \left[\int_\xi^x \left(\frac{u_1}{\delta_4}\right)^{1/2} dx\right]^{-1/3} \tag{6}$$

(Note that $(6{\cdot}41)^{1/3} \doteqdot (1/3)!\,3^{2/3}$, which is the form appearing in Lighthill's paper.)

2.4. *The Tribus–Klein–Tifford 'improvement'*

The Lighthill solution (5) or (6) is asymptotically exact, within the validity of the boundary-layer approximation, if $\Delta_4 \ll \delta_4$. It may give appreciable error in other cases. Such cases occur among the isothermal wedge solutions, when $\sigma \gg 1$ and the pressure gradient is finite. Noting this, Tribus & Klein (1955), following Tifford (1951), have proposed a correction to Lighthill's formula which depends on σ and the pressure gradient parameter alone. Their correction is such as to make the formula exact for all the isothermal wedges.

This procedure, however, is open to the following objections. Suppose that the heated section on a wedge does not start at the apex but much farther back. Then the thermal boundary layer is comparatively thin, the Lighthill expression (6) is accurate, and no correction is needed. Application of the Tribus–Klein–Tifford correction will therefore *introduce* an error of the same magnitude as that which it is intended to eliminate.

It is clear that if the Lighthill formula is to be improved, the correction procedure must take account of the relative thicknesses of the boundary layers. Yet how is such a correction to be derived from the isothermal wedge solutions if these have a thickness ratio uniquely determined by Prandtl number and pressure gradient parameter?—only by re-grouping the isothermal wedge solutions in the light of intuition.

2.5. *The new correction*

Suppose that the correction required, to account for the influence of Δ_4/δ_4, $(\delta_4^2/\nu)\,du_1/dx$, and σ, is a function solely of the extent of the thermal boundary layer into the region where the velocity profile in the boundary layer is curved. The latter quantity is measured by the dimensionless group $(\Delta_4\,\delta_4/\nu)\,du_1/dx$; for the velocity distribution close to the wall can be written

$$\begin{aligned} u &= \left(\frac{\partial u}{\partial y}\right)_{y=0} y + \frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} y^2 \\ &= \frac{u_1}{\delta_4} y - \frac{1}{2}\frac{u_1}{\nu}\frac{du_1}{dx} y^2, \end{aligned} \tag{7}$$

and at $y = \Delta_4$ the ratio of the quadratic to the linear term is

$$\left(\frac{-u_1}{2\nu}\,\frac{du_1}{dx}\,\Delta_4^2\right)\Big/\left(\frac{u_1\,\Delta_4}{\delta_4}\right) = -\frac{\Delta_4\,\delta_4}{2\nu}\,\frac{du_1}{dx}. \tag{8}$$

If the above supposition is correct, it ought to be possible to write the continuation equation (3) with high accuracy in the form

$$\frac{1}{\alpha}\left(\frac{\delta_4}{u_1}\right)^{1/2}\frac{d}{dx}\left[\Delta_4^3\left(\frac{u_1}{\delta_4}\right)^{3/2}\right] = 6{\cdot}41 + F\left(\frac{\Delta_4\,\delta_4}{\nu}\,\frac{du_1}{dx}\right), \tag{9}$$

in which the form of the left-hand side and the constant on the right-hand side derive from the Lighthill solution, and F is the as yet unknown correction term. This function may be expected to vanish with its argument.

The validity of equation (9) can be tested by reference to the isothermal wedge solutions; for if $(\Delta_4\,\delta_4/\nu)\,du_1/dx$ is as important as has been supposed, these solutions plotted in the form suggested by (9) should form a single curve instead of a one-parameter family of curves.

This is put to the test in figure 2 on which the isothermal wedge data of Eckert (1942) and of Livingood & Donoughe (1955) are plotted. It will be seen that the points indeed lie close to a single curve; the supposition is apparently well-founded.

Some scatter is apparent however. The worst is in the neighbourhood of $(\Delta_4\,\delta_4/\nu)\,du_1/dx = 0$, the case of zero pressure gradient. Here the isothermal wedge point for $\sigma = 0{\cdot}7$ lies at an ordinate of 6·9 instead of the expected 6·41. Since the heat transfer rate is proportional to the reciprocal cube root of the ordinate, the error in heat flux is only 2·5%, which may be considered acceptable. This is of course the same error as was noted by Lighthill for the case of the flat plate.

Use of equation (9) can therefore be expected to yield errors in heat flux of less than 2·5%, particularly when the heated section starts well back from the leading edge.

2.6. *Free convection and the rotating disc*

Also plotted on figure 2 are points obtained from solutions valid for free convection from a flat plate and for forced convection from a rotating disc. The former data have been derived from the papers of Schmidt & Beckmann (1930), Schuh (1948), Ostrach (1953), and Stewartson & Jones (1957). The latter data have been obtained by interpolation in small-scale graphs presented by Millsaps & Pohlhausen (1952); their accuracy is therefore poor. For both free convection and the rotating disc, the curvature

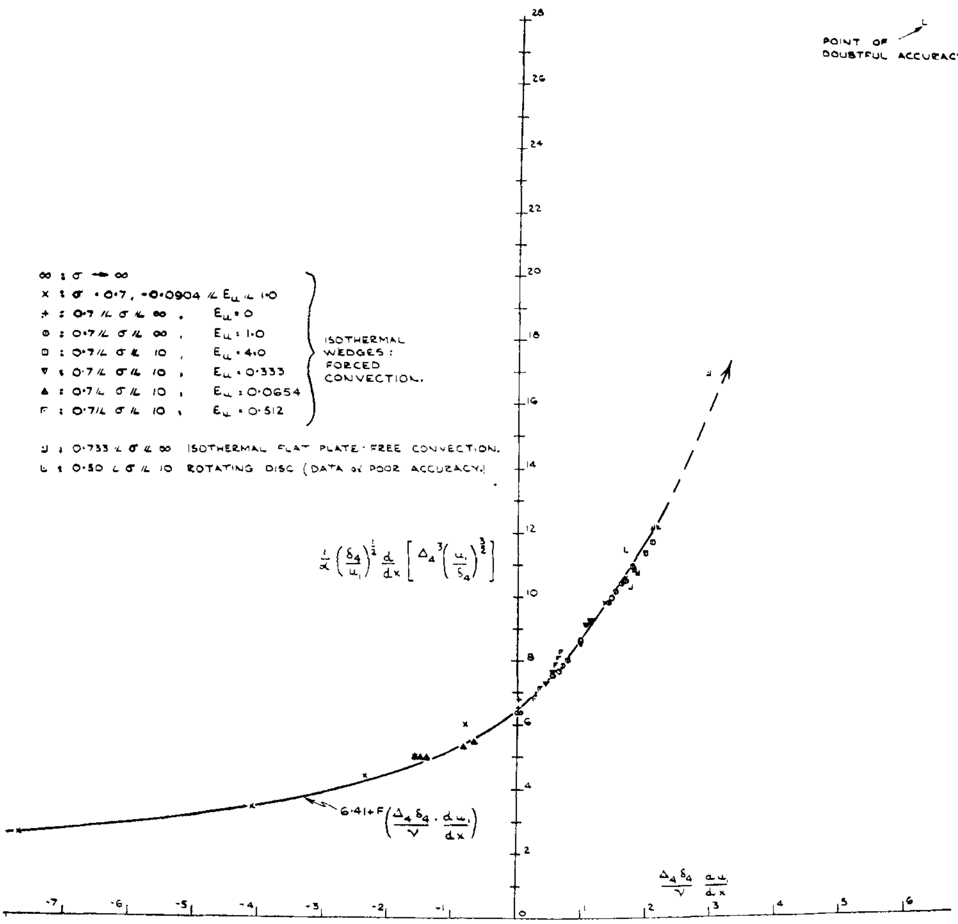

Figure 2. Deduction of the continuation function from the isothermal wedge solutions : conduction thickness Δ_4.

parameter in the abscissa of figure 2 is $-\Delta_4(\partial^2 u/\partial y^2)_{y=0}/(\partial u/\partial y)_{y=0}$, instead of the group $(\Delta_4 \delta_4/\nu)\, du_1/dx$ which is equivalent for the wedge solutions only. In evaluating the rotating-disc solutions, Mangler's (1948) transformation from rotationally-symmetric to plane coordinates was used.

The free-convection and rotating-disc points lie close to a prolongation of the curve for the forced-convection points. Their nearness is further confirmation of the validity of the curvature parameter. The fact that this has high positive values for these cases, even with very high Prandtl number, is a reflection of the fact that the boundary layers are highly accelerated, in the one case by buoyancy forces, in the other by centrifugal forces.

Figure 2 summarizes almost all the heat transfer solutions available in the literature for constant-property boundary-layer flow past an impermeable wall. It is evident that more solutions are needed before the field can be regarded as adequately covered. Nevertheless, the coordinates used in plotting figure 2, by bringing the points near to a single curve, greatly diminish the amount of further work which is needed. They may be expected to prove helpful also in presenting data for variable properties with a permeable wall.

2.7. *Integration*

Many numerical methods are available for integrating equations such as (9), in which it will be recalled that Δ_4 is the dependent variable, x is the independent one, and u_1, du_1/dx, and δ_4 are known functions of x. However, since F in (9) is normally small, the following quadrature (essentially a Picard approximation) will often suffice. The solution of (5) is written as

$$\left[\Delta_4^3\left(\frac{u_1}{\delta_4}\right)^{3/2}\right]_x - \left[\Delta_4^3\left(\frac{u_1}{\delta_4}\right)^{3/2}\right]_\xi = 6{\cdot}41\alpha\int_\xi^x\left(\frac{u_1}{\delta_4}\right)^{1/2}dx + \alpha\int_\xi^x\left(\frac{u_1}{\delta_4}\right)^{1/2}F\left(\Delta_4'\frac{\delta_4}{\nu}\frac{du_1}{dx}\right)dx, \quad (10)$$

the subscripts to the brackets denoting the position at which they are evaluated.

Written explicitly for the required quantity $\Delta_4(x)$, (10) becomes (for the case when $\Delta_4 = 0$ at $x = \xi$, as is usual)

$$\Delta_4 = \left(\frac{\delta_4}{u_1}\right)^{1/2}\left[6{\cdot}41\alpha\int_\xi^x\left(\frac{u_1}{\delta_4}\right)^{1/2}dx + \alpha\int_\xi^x\left(\frac{u_1}{\delta_4}\right)^{1/2}F\left(\frac{\Delta_4'\delta_4}{\nu}\frac{du_1}{dx}\right)dx\right]^{1/3}. \quad (11)$$

In (10) and (11), Δ_4' appearing in the argument of F is the first approximation for Δ_4, obtained from (11) by omitting the second integral. Values of F may be read from the curve drawn in figure 2. If in a particular case the second integral appears to be important, a second approximation can obviously be made.

2.8. *Example*

A calculation of the distribution of heat flux from an ellipse, of axis ratio 1 : 2 and uniform temperature (the case that was first studied by Eckert (1942)) has been made by the above method. The result, in non-dimensional form, is represented in figure 3, where it may be compared with the calculation by Eckert's class I procedure. The present results lie somewhat below those of Eckert, but it should not be supposed that they are therefore less accurate; for class I procedures involve graver approximations than the present class II method. No experimental results of sufficient accuracy are available for comparison.

It should be noted that in the example, in which the heated section starts at a stagnation point, the integration has to be begun by a straightforward application of L'Hopital's rule.

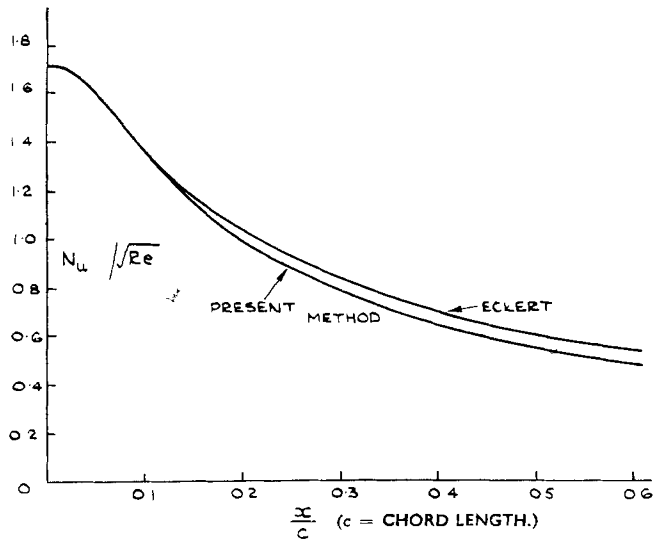

Figure 3. Calculated local heat flux distribution on ellipse of axis ratio 1 : 2.

3. Determination of total heat transfer rate by use of the enthalpy-flux thickness Δ_2

When the local heat transfer has been determined by the method of the last section, the total heat transfer rate can be found by integration. If only the total heat transfer is required, a quicker method is to evaluate $u_1 \Delta_2$ directly and apply equation (2). A continuation equation for Δ_2 can be set up by a combined use of dimensional analysis, intuition, and the exact similar solutions, as before. The solution for the case treated by Lighthill may be written

$$\frac{1}{\alpha}\left(\frac{\delta_4}{u_1}\right)^{1/2} \frac{d}{dx}(u_1 \Delta_2)^{3/2} = 0{\cdot}725. \tag{12}$$

The appropriate 'curvature parameter' is $(\Delta_2^{1/2}\delta_4^{3/2}/\nu)\, du_1/dx$. The wedge solutions for various Prandtl numbers have been plotted in the corresponding form in figure 4, which confirms expectations by yielding a grouping of the points close to a single curve. Fortuitously, this curve is very nearly a straight line, so that an approximate linear expression can be found. The continuation equation thus becomes

$$\frac{1}{\alpha}\left(\frac{\delta_4}{u_1}\right)^{1/2} \frac{d}{dx}(u_1 \Delta_2)^{3/2} = 0{\cdot}7 - 0{\cdot}1\,\frac{\Delta_2^{1/2}\delta_4^{3/2}}{\nu}\,\frac{du_1}{dx}. \tag{13}$$

Comparison of (12) and (13) for the case $du_1/dx = 0$ shows the order of magnitude of the maximum error involved in the linear approximation, and indeed in the method as a whole: about 2·5% in the solution for heat flow.

Equation (13) can be integrated as a quadrature with a Picard approximation as before. The required form is

$$u_1\Delta_2 = \left[0{\cdot}7\alpha\int_\xi^l \left(\frac{u_1}{\delta_4}\right)^{1/2} dx - \frac{0{\cdot}1}{\sigma}\int_\xi^x (u_1\Delta_2')^{1/2}\delta_4 \frac{du_1}{dx}\,dx\right]^{2/3}, \qquad (14)$$

wherein Δ_2' is the first approximation for Δ_2, obtained from (14) by omitting the second integral. The curve on figure 4 is so nearly horizontal that a second approximation will rarely be worth while.

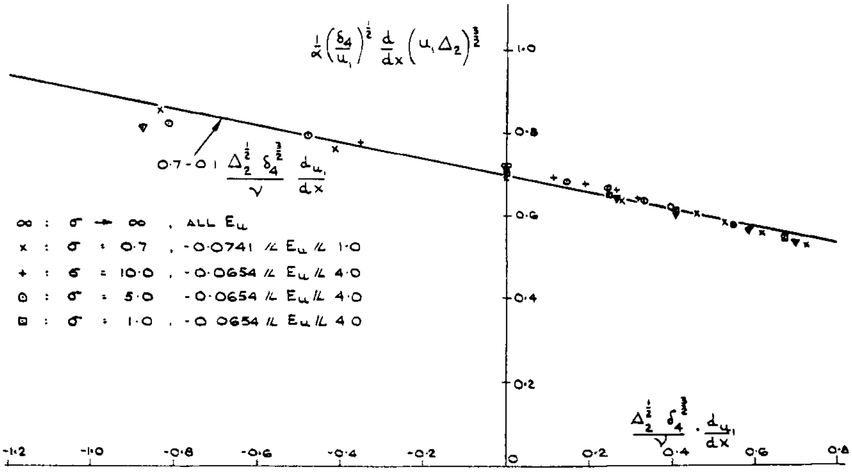

Figure 4. Deduction of the continuation function from the isothermal wedge solutions : enthalpy–flux thickness Δ_2.

4. Heat transfer from a surface of varying temperature

If the wall temperature T_0 varies with ξ in a known way, the local heat transfer at a point x is determined by superposition in the manner already indicated by Lighthill. A brief summary is given here. The effects on the heat transfer at x of each increment of wall temperature dT_0 at ξ are added by the (Stieltjes) integral

$$\dot{q}''(x) = k\int_0^x \frac{1}{\Delta_4(\xi, x)}\,dT_0(\xi), \qquad (15)$$

where $\Delta_4(\xi, x)$ for each ξ and the appropriate x is obtained from (7). To evaluate (15), x is regarded as fixed in the integration and $dT_0(\xi)$ is written as $(dT_0/d\xi)\,d\xi$, except at $\xi = 0$ and at discontinuous jumps of wall temperature. The wall temperature gradient $dT_0/d\xi$ is supposed given.

The total heat flow from the aerofoil is evaluated in a similar way from the integral

$$\dot{q}' = cp\int_0^l u_1\Delta_2(\xi, l)\,dT_0(\xi). \qquad (16)$$

5. Determination of u_1/δ_4 as a function of x

Finally the distribution of u_1/δ_4, which has been assumed known, will be discussed. Tribus & Klein (1955) advocate solving a continuation equation for the momentum thickness numerically by the isocline method. Simpler methods exist, however: for example, the approximate quadratures introduced by Walz (1941), Thwaites (1949), and others. A more exact technique similar to that used above for heat transfer will be mentioned here.

Dimensional analysis and the boundary-layer assumption yield a continuation equation for the momentum thickness δ_2 in which the unknown function can be obtained from the wedge solutions. A convenient form of the equation is found to be

$$\frac{1}{\nu u_1^{4\cdot 17}}\frac{d}{dx}(u_1^{5\cdot 17}\delta_2^2) = 0{\cdot}4418 - f\left(\frac{\delta_2^2}{\nu}\frac{du_1}{dx}\right), \tag{17}$$

for then the second-term on the right-hand side is very small. Table 2 contains a few values, computed from Hartree's solution for the wedge boundary layers (Hartree, 1937).

$\dfrac{\delta_2^2}{\nu}\dfrac{du_1}{dx}$	$-0{\cdot}0682$	$-0{\cdot}0266$	0	$0{\cdot}0333$	$0{\cdot}0611$	$0{\cdot}0855$
$f\left(\dfrac{\delta_2^2}{\nu}\dfrac{du_1}{dx}\right)$	$-0{\cdot}026$	$-0{\cdot}0058$	0	$0{\cdot}0033$	$0{\cdot}0019$	0
$(\delta_4/\delta_2)^2$	∞	$36{\cdot}9$	$20{\cdot}5$	$12{\cdot}75$	$9{\cdot}50$	$7{\cdot}70$

Table 2.

Equation (17) can be integrated in the manner used for equation (9) and (13), a first approximation for δ_2^2 being used in the argument of f. A second approximation will hardly ever be worth while.

Once δ_2, the momentum thickness, has been calculated as a function of x, the hypothesis that only members of the wedge family of velocity profiles occur is invoked for the calculation of corresponding values of the shear thickness δ_4. For δ_4/δ_2 is known for wedges; it is a unique function of the pressure gradient parameter, $\delta_2^2(du_1/dx)/\nu$. Table 2 gives some values based on the work of Hartree. Consultation of a more extended table of this nature gives δ_4 as a function of x, and so enables the heat transfer calculation to proceed.

6. Conclusions

1. The Lighthill method of calculating heat transfer in laminar flow has been improved by

(*a*) a better allowance than that of Tifford for the non-linearity of the velocity profile;

(*b*) the use of the enthalpy-flux thickness for calculating the heat transfer over a finite area of the surface; and

(*c*) a quadrature procedure for integrating the continuation equation for the boundary layer thicknesses.

2. The effects of pressure gradient and Prandtl number on the rate of growth of the thermal boundary layer can be described by a single 'curvature parameter' which expresses the extent of the temperature boundary layer into the region where the velocity profile curvature is noticeable.

3. The conclusion 2 is supported by examination of the wedge solutions of Eckert and others.

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References

Eckert, E. R. G. 1942 *V.D.I. Forschungsheft* **416.**

Hartree, D. R. 1937 *Proc. Camb. Phil. Soc.* **33,** 223.

Lighthill, M. J. 1950 *Proc. Roy. Soc.* A, **202,** 359.

Livingood, J. N. B., & Donoughe, P. L. 1955 *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 3588.

Mangler, W. 1948 *Z. angew. Math. Mech.* **28,** 97.

Millsaps, K. & Pohlhausen, K. 1952 *J. Aero. Sci.* **19,** 120.

Ostrach, S. 1953 *Nat. Adv. Comm. Aero., Wash., Rep.* no. 1111.

Schmidt, E., & Beckmann, W. 1930 *Tech. Mech. u. Thermodynamik* (*Forschung. Ing.-Wes.*) **1,** 391.

Schuh, H. 1948 Boundary layers of temperature. *Ministry of Supply German Document Centre*, no. 3220T.

Schuh, H. 1953 *K. T. H., Stockholm, Aero T. N.*, 33.

Smith, A. G., & Spalding, D. B. 1958 *J. Roy. Aero. Soc.* **62,** 60.

Spalding, D. B. & Smith, A. G. 1958 Verbrennung flüssiger und fester Brennstoffe, ein Grenzschichtproblem. (To be published in *Brennstoff-Wärme-Kraft*.)

Squire, H. B. 1942 *Aero. Res. Counc., Lond., Rep. & Mem.*, no. 1986.

Stewartson, K. & Jones, L. T. 1957 *J. Aero. Sci.* **24,** 379.

Tifford, A. N. 1951 *J. Aero. Sci.* **18,** 283.

Thwaites, B. 1949 *Aero. Quart.* **1,** 245.

Tribus, M., & Klein, J. 1955 *J. Aero. Sci.* **22,** 62.

Walz, A. 1941, *Lilienthal Bericht* **141,** 8.